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Complex dynamical systems of the quartic polynomials

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Abstract

The space $\mathrm{M}_4(\mathbb{C})$ is the space of all affine conjugacy classes of quartic polynomials. We define a projection Ψ_4 from this space to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In [2], we show the projection is not surjective. The image of $\mathrm{M}_4(\mathbb{C})$ under Ψ_4 is denoted by $\Sigma(4)$. The complement $\mathbb{C}^3 \setminus \Sigma(4)$ is called the exceptional set. By analyzing the dynamics on the section $\{(4, \sigma_2, \sigma_4)\}$, we verify that quartic polynomial degenerates into "twins" of quadratic polynomials on the exceptional set.

1 Introduction

Let $\mathrm{Poly}_4(\mathbb{C})$ be the space of all quartic polynomials, and $\mathrm{M}_4(\mathbb{C})$ be the space of all affine conjugacy classes of quartic polynomials. We define a projection Ψ_4 from $\mathrm{M}_4(\mathbb{C})$ to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In [2], we show the projection is not surjective. The image of $\mathrm{M}_4(\mathbb{C})$ under Ψ_4 is denoted by $\Sigma(4)$. The complement $\mathbb{C}^3 \setminus \Sigma(4)$ is denoted by $\mathcal{E}(4)$, and called the exceptional set. For the cubic (resp. quadratic) polynomials, the exceptional set is empty.

As a Corollary of Theorem 1 in [3] we have: *If n given values $m_1, m_2, \cdots, m_n$ satisfy $\sum_{i=1}^{n} \frac{1}{1-m_i} = 0$ and if $\sum_{j=1}^{k} \frac{1}{1-m_{i_j}} \neq 0$ for any choice of $\{i_j\}_{j=1}^{k}, 1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$, then there exists a polynomial of degree exactly n having the fixed points of the multipliers $m_1, m_2, \cdots, m_n$.*

We define an algebraic variety, $G(c)$ defined in Section 2, that indicates essential property of the projection Ψ_4, and as Theorem 1 we have a defining equation of the exceptional set and of the branch locus.

According to Theorem 1, we will need to consider the following:

- Why the exceptional set is non empty?
- Find a relation between dynamics of conjugacy classes in $\Psi_4^{-1}(s)$, $s \in \mathbb{C}^3$.

In this paper, we examine dynamical behavior on the parameter space $\Sigma(4) \cup \mathcal{E}(4)$ (disjoint union), and we have the following conjectures by constructing of two suitable polynomial-like maps.

Conjecture On the exceptional set, a quartic polynomial degenerates into "twins" of quadratic polynomials conjugate to z^2+c for some c.

Conjecture None of quartic polynomial p has two disjoint quadratic-like restrictions of p such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial $z^2+c,\ c\in \mathrm{M}\setminus\{\frac{1}{4}\}$, where M is Mandelbrot set.

These conjectures give the reason why the exceptional set is not empty. The following theorem gives a support for these conjectures.

Theorem There is a component $D\subset\Sigma(4)$ such that two polynomial-like maps $(U,V,p)\sim_{hb} z^2+c$ and $(\tilde{U},\tilde{V},p)\sim_{hb} z^2+\bar{c}$ are constructed for any $\langle p\rangle\in D$, and the imaginary part of c converges to zero as $\langle p\rangle\to\mathcal{E}(4)$.

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2 Definitions

2.1 Definitions and Notations

Let $\mathrm{Poly}_4(\mathbb{C})$ be the space of all polynomials of the form

$$\begin{aligned}&p:\mathbb{C}\to\mathbb{C},\\&p(z)=a_4z^4+a_3z^3+a_2z^3+a_1z+a_0\quad(a_4\neq 0).\end{aligned}$$

Two maps $p_1,p_2\in\mathrm{Poly}_4(\mathbb{C})$ are *holomorphically conjugate*, denoted by $p_1\sim p_2$, if and only if there exists $g\in\mathfrak{A}(\mathbb{C})$ with $g\circ p_1\circ g^{-1}=p_2$, where $\mathfrak{A}(\mathbb{C})$ is the group of all affine transformations.

The space, $\mathrm{Poly}_4(\mathbb{C})/_\sim$, of holomorphic conjugacy classes $\langle p\rangle$ of quartic polynomials is denoted by $\mathrm{M}_4(\mathbb{C})$.

For each $p(z)\in\mathrm{Poly}_4(\mathbb{C})$, let $z_1,\ \cdots,\ z_4,\ z_5=\infty$ be the fixed points of p, and $\mu_1,\ \cdots,\ \mu_4,\ \mu_5=0$ the multipliers of z_i (i.e. $\mu_i=p'(z_i)$).
Let $\sigma_1,\sigma_2,\cdots,\sigma_5$ be the elementary symmetric functions of these multipliers

$$\begin{aligned}\sigma_1&=\mu_1+\mu_2+\mu_3+\mu_4,\\\sigma_2&=\mu_1\mu_2+\mu_1\mu_3+\mu_1\mu_4+\mu_2\mu_3+\mu_2\mu_4+\mu_3\mu_4,\\\sigma_3&=\mu_1\mu_2\mu_3+\mu_1\mu_2\mu_4+\mu_1\mu_3\mu_4+\mu_2\mu_3\mu_4,\\\sigma_4&=\mu_1\mu_2\mu_3\mu_4,\\\sigma_5&=0.\end{aligned}$$

These multipliers are *invariant* under the action of (conjugation) $\mathfrak{A}(\mathbb{C})$.

The holomorphic index of a rational function f at a fixed point $\zeta\in\mathbb{C}$ is defined to be the complex number

$$\iota(f,\zeta)=\frac{1}{2\pi i}\oint\frac{dz}{z-f(z)},$$

where we integrate in a small loop in the positive direction around ζ.

The following results are well known as "Fatou's index theorem":

- If a multiplier $\mu \neq 1$, then $\iota(f,\zeta) = \frac{1}{1-\mu}$.
- For any polynomial p which is not the identity map,

$$\sum_{\zeta\in\mathbb{C}} \iota(p,\zeta) = 0, \tag{1}$$

where this summation is over all fixed points of p.

A *polynomial-like* map of degree d is a triple (U,V,f) where U and V are topological disks, with V relatively compact in U, and $f : V \to U$ is analytic, proper of degree d.

The filled-in Julia set K_f of a polynomial-like map (U,V,f) is defined by

$$K_f = \bigcap_{n\geq 0} f^{-n}(V).$$

Polynomial-like maps (U,V,f) and $(\widetilde{U},\widetilde{V},\tilde{f})$ are *hybrid equivalent*, $f \sim_{hb} \tilde{f}$, if there exists a quasi-conformal homeomorphism h from a neighborhood of K_f to a neighborhood of $K_{\tilde{f}}$, such that $h\circ f = \tilde{f}\circ h$ near K_f and $\bar{\partial}h = 0$ almost everywhere on K_f.

From Straightening Theorem in [1], every polynomial-like map (U,V,f) of degree d is hybrid equivalent to a polynomial P of degree d. If K_f is connected then P is unique up to conjugation by an affine map.

2.2 Transformation formula

The following relation is obtained by Fatou's index theorem.

Lemma 1 (Theorem 1 in [2]) Among $\sigma_i{}'s$, there is a linear relation

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0.$$

For a monic and centered quartic polynomial $z^4+c_2z^2+c_1z+c_0$, the three values σ_1, σ_2, σ_4 are given by **Transformation formula**:

$$\begin{aligned}
\sigma_1 &= -8c_1 + 12,\\
\sigma_2 &= 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48,\\
\sigma_4 &= 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + (144c_0c_1^2 - 288c_0c_1 + 128c_0)c_2\\
&\quad -27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3.
\end{aligned}$$

To remove an affine ambiguity from Transformation formula, we consider the following:

1. for a point $\langle p\rangle \in \mathrm{M}_4(\mathbb{C})$, choose a monic and centered representative $z^4+c_2z^2+c_1z+c_0$.
2. getting rid of the affine ambiguity on "Transformation formula", set $c := c_2^3$ (if $c_2 = 0$, set $\tilde{c} := c_0^3$), and
3. rebuild Transformation formula of $\sigma_1, \sigma_2, \sigma_4, c, c_0, c_1$ variables.

4. remove two variables c_0, c_1, from the above formula.

After these procedure, we obtain a parametrized algebraic variety.

Definition We define an algebraic variety in $\mathbb{C}^3$ with a parameter $c \in \mathbb{C}$,

$$\begin{aligned} G(c):\quad & 262144(\sigma_1-4)^2c^2+1024(27\sigma_1^4+(-144\sigma_2-576)\sigma_1^2+(384\sigma_2+1280)\sigma_1+128\sigma_2^2 \\ & -256\sigma_2-512\sigma_4-768)c+(9\sigma_1^2+24\sigma_1-32\sigma_2-48)^3=0. \end{aligned}$$

$G(c)$ implies the following: For any point $(\sigma_1,\sigma_2,\sigma_4) \in \mathbb{C}^3$, on $G(c)$, the number of parameter values is equal to the number of conjugacy classes corresponds to the point $(\sigma_1,\sigma_2,\sigma_4)$.

Hence, there is a natural projection

$$\begin{array}{cccc} \Psi_4: & \mathrm{M}_4(\mathbb{C}) & \longrightarrow & \Sigma(4) \\ & \Psi & & \Psi \\ & \langle p \rangle & \longmapsto & (\sigma_1,\sigma_2,\sigma_4), \end{array}$$

where $\Sigma(4)$ is the image of $\mathrm{M}_4(\mathbb{C})$ under Ψ_4. The complement $\mathbb{C}^3 \setminus \Sigma(4)$ is denoted by $\mathcal{E}(4)$, and called the *exceptional set*.

The algebraic variety $G(c)$ perfectly exhibits phenomena induced by $\Psi_4 : \mathrm{M}_4(\mathbb{C}) \to \Sigma(4)$. Therefore we have the following Theorem.

Theorem 1 For $(\sigma_1,\sigma_2,\sigma_4) \in \mathbb{C}^3$, number of the elements of set $\Psi_4^{-1}(\sigma_1,\sigma_2,\sigma_4)$ are ∞, 0, 1 or 2.

Case 1 $\#\Psi_4^{-1}(\sigma_1,\sigma_2,\sigma_4)=\infty$ if and only if $(\sigma_1,\sigma_2,\sigma_4)=(4,6,1)$.

$$\Psi_4^{-1}(4,6,1)=\{p_a(z)=(z^2-a)^2+z\}_{a\in\mathbb{C}} \quad (\text{note } p_a \sim p_{\pm\omega a} \text{ by } z \mapsto \pm\omega z)$$

Case 2 $\#\Psi_4^{-1}(\sigma_1,\sigma_2,\sigma_4)=0$ if and only if the point $(\sigma_1,\sigma_2,\sigma_4)$ cannot belong to $G(c)$ for any c.

$$(\sigma_1,\sigma_2,\sigma_4)=\left(4,\ s,\ \frac{(s-4)^2}{4}\right), \quad s \neq 6. \ \textbf{(the exceptional set)}$$

Case 3 $\#\Psi_4^{-1}(\sigma_1,\sigma_2,\sigma_4)=1$ if and only if discriminant of the defining equation of $G(c)$ vanishes or $\sigma_1=4$ (**the branch locus**).

Case 4 $\#\Psi_4^{-1}(\sigma_1,\sigma_2,\sigma_4)=2$, for the remains of the above.

Theorem 1 leads immediately to the following two corollaries.

Corollary 1 The exceptional set $\mathcal{E}(4)$ is contained in the plane $\{(4,\sigma_2,\sigma_4)\} \cong \mathbb{C}^2$.

Corollary 2 There is not a quartic polynomial having the fixed points of the multipliers $\mu,\mu,2-\mu,2-\mu$, $(\mu \neq 1)$.

3 Loci $\mathrm{Per}_1(\mu)$ on the space $\{(4, s_2, s_4)\}$

In this section, we consider dynamical behavior on the real section $\mathbb{R}^2 \cong \{(4, s_2, s_4)\}$, by Theorem 1, and show some figures supporting the conjectures.

The locus $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle p \rangle$ of maps p having a fixed point of multiplier μ.

Proposition 1 For each $\mu \in \mathbb{C}$, $\mathrm{Per}_1(\mu)$ is a straight line with the following defining equation:

$$\mathrm{Per}_1(\mu) : \ \sigma_4 - (2\mu - \mu^2)\sigma_2 + \mu^4 - 4\mu^3 + 8\mu = 0.$$

Proof. The multipliers at the fixed points are the roots of the equation,

$$\mu^4 - \sigma_1\mu^3 + \sigma_2\mu^2 - \sigma_3\mu + \sigma_4 = 0.$$

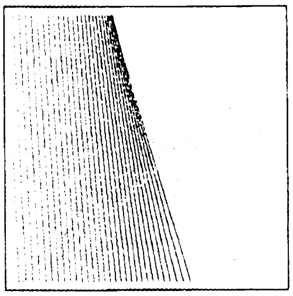
From the linear relation of Lemma 1, we have the defining equation of $\mathrm{Per}_1(\mu)$. ∎

We remark that the cases of the multipliers of a quartic polynomial on the real plane $\{(4, \sigma_2, \sigma_4)\}$ are 'four real values', 'two real and a pair of complex conjugates', or 'two pair of complex conjugates'.

3.1 $\mathrm{Per}_1(\mu)$ $(\mu \in \mathbb{R})$

At first we consider $\mu \in \mathbb{R}$. In this case we can illustrate the figure of $\mathrm{Per}_1(\mu)$. (See Figure 1.) The following results are easily verified.

Proposition 2 For $\langle p \rangle \in \{(4, \sigma_2, \sigma_4)\} \cap \Sigma(4)$, the corresponding multipliers of p are $\mu, 2 - \mu, \lambda, 2 - \lambda$.



The left figure shows $\mathrm{Per}_1(\mu)$ $(-10 < \mu < 1)$:
$-20 < s_2, s_4 < 20$,
Gray lines mean $\mathrm{Per}_1(\mu)$ $(|\mu| \geq 1)$ and
black lines mean $\mathrm{Per}_1(\mu)$ $(|\mu| < 1)$.

Figure 1:

Corollary 3

- If p has a attracting fixed point then p has a repelling fixed point with positive multiplier.
- If p has a repelling fixed point with negative multiplier then p has a repelling fixed point with positive multiplier.

Namely, each line of Figure 1 is overlapped by a line $\mathrm{Per}_1(\mu)$ for some $\mu > 1$, and p cannot have three attracting fixed points.

3.2 $\mathrm{Per}_1(\mu)$ and $\mathrm{Per}_1(\bar{\mu})$

Next, we consider the multipliers of a quartic polynomial are 'two real and a pair of complex conjugates'. In this case, the multipliers are $1 \pm i\beta$, λ, and $2-\lambda$ from Proposition 2. Then we have the following from Proposition 1.

Proposition 3 For each $\beta \in \mathbb{R}$, $\mathrm{Per}_1(1 \pm i\beta)$ is a straight line with the following defining equation:

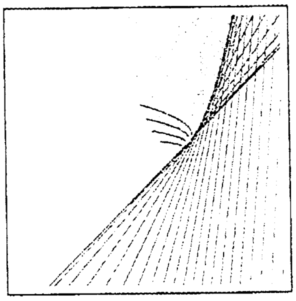
$$\mathrm{Per}_1(1 \pm i\beta): \quad \sigma_4 = (1+\beta^2)\sigma_2 - (1+\beta^2)(5+\beta^2).$$

Proof. Removing λ from two equations $\sigma_2 = 5 + \beta^2 + \lambda(2-\lambda)$ and $\sigma_4 = (1+\beta^2)\lambda(2-\lambda)$, we have the above defining equation of $\mathrm{Per}_1(1 \pm i\beta)$. ∎

Note that these loci are corresponds to repelling fixed points.

Now, we consider the last case: multipliers of a quartic polynomial are 'two pair of complex conjugates'. In this case, the multipliers are $a \pm ib$ and $2 - a \pm ib$ from Proposition 2. Because defining equation of $\mathrm{Per}_1(\mu)$ can express a line on the real plane no longer, we need a new device $\widetilde{\mathrm{Per}}_1(t)$ for illustrating figures of $\mathrm{Per}_1(\mu)$. (See Figure 2.)

The locus $\widetilde{\mathrm{Per}}_1(t)$ be the set of all conjugacy classes $\langle p \rangle$ of maps p having a fixed point of multiplier μ with $t = \mu\bar{\mu}$.



The left figure shows $\mathrm{Per}_1(1 \pm i\beta)$ and $\widetilde{\mathrm{Per}}_1(t)$.
$-20 < s_2, s_4 < 20$,
Dark gray lines mean $\mathrm{Per}_1(1 \pm i\beta)$,
gray curves mean $\widetilde{\mathrm{Per}}_1(t)$, $t \geq 1$ and
black curves mean $\widetilde{\mathrm{Per}}_1(t)$, $t < 1$.

Figure 2:

Proposition 4 In the case that the multipliers are $a \pm ib$ and $2 - a \pm ib$, we have a defining equation of $\widetilde{\mathrm{Per}}_1(t)$.

$$\widetilde{\mathrm{Per}}_1(t): \quad \sigma_4^2 - 2(t^2+2t)\sigma_4 + t^4 - 4t^3 + (\sigma_2 - 16)t^2 = 0,$$

where $t = a^2 + b^2$.

Proof. In this case the multipliers are $a \pm ib$ and $2 - a \pm ib$. By setting $t = a^2 + b^2$ for two equations $\sigma_2 = -2a^2 + 4a + 4 + 2b^2$ and $\sigma_4 = (a^2+b^2)((2-a)^2+b^2)$, we have

$$\sigma_2 = -4a^2 + 4a + 4 + 2t, \qquad \sigma_4 = t(t - 4a + 4). \tag{2}$$

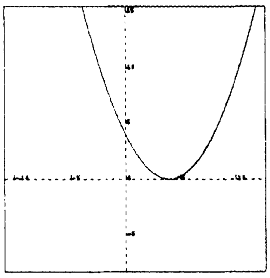
Removing a from the above two equations, we have a defining equation of $\widetilde{\mathrm{Per}}_1(t)$. ∎

Remark If $0 \le t < 1$, $\widetilde{\mathrm{Per}_1}(t)$ corresponds to polynomials having two attracting fixed points of multiplier $a+ib$ and $a-ib$. As $a, b \in \mathbb{R}$, the discriminant $4+4(4+2t-\sigma_2)$ of (2) must be positive. Therefore, on a region $\{(4,\sigma_2,\sigma_4)\,|\,\sigma_2 < -\frac{1}{4}(\sigma_4^2 - 6\sigma_4 - 19),\ \sigma_4 < \frac{(2-\sigma_2)^2}{4}\}$, corresponding polynomial p have two attracting fixed points of multipliers $a \pm ib$.

4 The exceptional set

The lines $\{\mathrm{Per}_1(\mu)\}$ have a close relation with the exceptional set. As an example, we give the following results directly obtained by the results in the section 3.1 and 3.2.

- On the plane $\{(4, s_2, s_4)\} \cong \mathbb{R}^2$, the envelopes of the lines $\{\mathrm{Per}_1(\mu)\}_{\mu\in\mathbb{R}}$ and of $\{\mathrm{Per}_1(1 \pm i\beta)\}_{\beta\in\mathbb{R}}$ coincides with the exceptional set. (See Figure 1, 2 and 3.)
- On the region $\left\{(4,\sigma_2,\sigma_4)\,|\,\sigma_4 < \frac{(2-\sigma_2)^2}{4}\right\}$ that bounded by the exceptional set, corresponding quartic polynomial has the fixed points of the multiplier with two pair of complex conjugates.



The left figure shows the real section of the exceptional set

$$\mathcal{E}(4):\ \left(4,\ s,\ \frac{(s-4)^2}{4}\right),\quad (s \ne 6).$$

Figure 3:

Conjecture On the exceptional set, a quartic polynomial degenerates into "twins" of quadratic polynomials conjugate to $z^2 + c$ for some c.

Theorem 2 There is a component $D \subset \Sigma(4)$ such that two polynomial-like maps $(U, V, p) \sim_{hb} z^2+c$ and $(\tilde{U}, \tilde{V}, p) \sim_{hb} z^2+\bar{c}$ are constructed for any $\langle p\rangle \in D$, and the imaginary part of c converges to zero as $\langle p\rangle \to \mathcal{E}(4)$.

Proof. On a region $\{(4,\sigma_2,\sigma_4)\,|\,\sigma_2 < -\frac{1}{4}(\sigma_4^2 - 6\sigma_4 - 19),\ \sigma_4 < \frac{(2-\sigma_2)^2}{4}\}$, any corresponding polynomial $p(z)$ has two attracting fixed points of multiplier μ, $\bar{\mu}$. Dynamics of $p(z)$ are symmetry for the real axis. (See Figure 4.) Therefore we can choose suitable topological disk U, $\widetilde{U}$ bounded by equipotential curves such that (U, V, p) and $(\widetilde{U}, \widetilde{V}, p)$ $(U \cap \widetilde{U} = \emptyset)$ are quadratic-like maps hybrid equivalent to $z^2 + c$ and $z^2 + \bar{c}$ respectively. (See Figure 6 and 7.) ∎

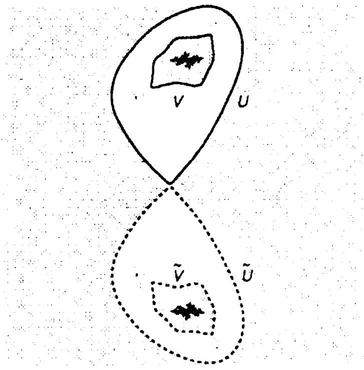


Figure 4: $(4, -1.7696160, 8.8480801)$, Julia set of $p(z) = z^4 + 3.8199z^2 + z + 3.775218$, $-2 < \Re z,\ \Im z < 2$

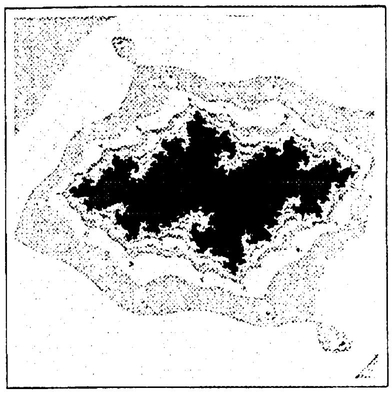


Figure 5: Julia set of $p(z) = z^4 + 3.8199z^2 + z + 3.775218$, $-0.2 < \Re z < 0.28$, $1.137 < \Im z < 1.617$

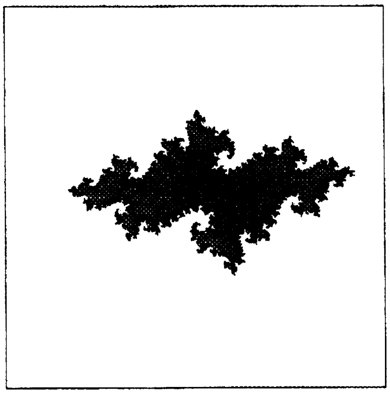


Figure 6: Julia set of quadratic-like map $-0.2 < \Re z < 0.28$, $1.137 < \Im z < 1.617$

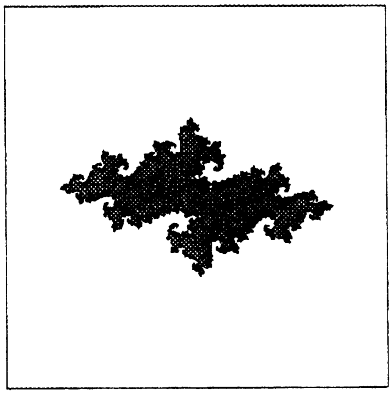


Figure 7: Julia set of $p_c(z) = z^2 + (-0.726 + 0.183i)$, $-2 < \Re z,\ \Im z < 2$.

5 On the point $(4, 6, 1) \in \Sigma(4)$

One parameter family $\{p_a(z) = (z^2 - a)^2 + a\}_{a \in \mathbb{C}}$ (note $p_a \sim p_{\pm\omega a}$ by $z \mapsto \pm\omega z$) corresponds to the point $(4, 6, 1)$. (See Figure 8 and 9.) There is a map p in this family such that p has two disjoint quadratic-like restriction hybrid equivalent to common quadratic map $z^2 + \frac{1}{4}$. (See Figure 8.)

Conjecture None of quartic polynomial p have two disjoint quadratic-like restrictions of p such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial $z^2 + c$, $c \in \mathrm{M} \setminus \{\frac{1}{4}\}$, where M is Mandelbrot set.

This conjecture gives a reason why the exceptional set is not empty.

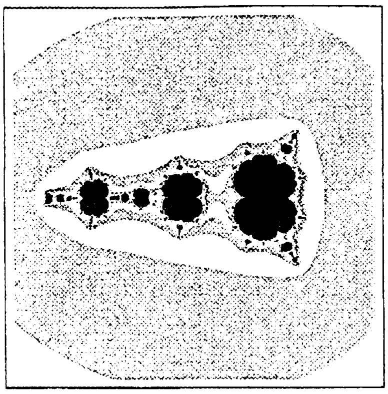


Figure 8: Julia set of $p(z) = z^4 - 2z^2 + z + 1$, $-2 < \Re z,\ \Im z < 2$. $(4,6,1) \in \Sigma(4)$

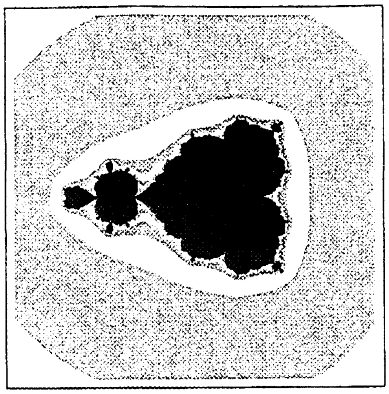


Figure 9: Julia set of $p(z) = z^4 - z^2 + z + 0.25$, $-2 < \Re z,\ \Im z < 2$. $(4,6,1) \in \Sigma(4)$

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